

# Locating-Dominating Sets and Identifying Codes of a Graph Associated to a Finite Vector Space

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## Abstract

In this paper, we investigate the problem of covering the vertices of a graph associated to a finite vector space as introduced by Das [16], such that we can uniquely identify any vertex by examining the vertices that cover it. We use locating-dominating sets and identifying codes, which are closely related concepts for this purpose. These sets consist of a dominating set of graph such that every vertex is uniquely identified by its neighborhood within the dominating sets. We find the location-domination number and the identifying number of the graph and study the exchange property for locating-dominating sets and identifying codes.

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## 1 Preliminaries

The association of graphs to algebraic structures has become the interesting research topic for the past few decades. See for instance: commuting graphs for groups [3, 7, 23], power graphs for groups and semigroups [9, 11, 30], zero divisor graph associated to a commutative ring [1, 4]. The association of a graph and vector space has history

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back in 1958 by Gould [19]. Later, Chen [14] investigated on vector spaces associated with a graph. Carvalho [10] has studied vector space and the Petersen Graph. In the recent past, Manjula [29] used vector spaces and made it possible to use techniques of linear algebra in studying the graph. Intersection graphs assigned to vector space were studied [24, 37]. Das [16] introduced a new graph structure, called non-zero component graph on finite dimensional vector spaces. He showed that the graph is connected and found its domination number and independence number [17]. He characterized the maximal cliques in the graph and found the exact clique number, for some particular cases [17]. Das has also given some results on size, edge-connectivity and the chromatic number of the graph [17].

The covering code problem for a given graph involves finding a minimum set of vertices whose neighborhoods uniquely overlap at any given graph vertex. The problem has demonstrated its fundamental nature through a wide variety of applications. Locating-dominating sets were introduced by Slater [34, 36] and identifying codes by Karpovsky et al. [25]. Locating-dominating sets are very similar to identifying codes with the subtle difference that only the vertices not in the locating-dominating set are required to have unique identifying sets. The decision problem for locating-dominating sets for directed graphs has been shown to be an NP-complete problem [12]. A considerable literature has been developed in this field (see [6, 13, 15, 18, 22, 31, 34, 35]). In [8], it was pointed out that each locating-dominating set is both locating and dominating set. However, a set that is both locating and dominating is not necessarily a locating-dominating set.

The initial application of locating-dominating sets and identifying codes was fault-diagnosis in the maintenance of multiprocessor systems [25]. More recently, identifying codes and locating-dominating sets were extended to applications for joint monitoring and routing in wireless sensor networks [28] and environmental monitoring [5].

A natural question arises in reader's mind that how can we distinguish the need of identifying codes or locating-dominating sets for a system? A system, in which processors or sensors are able to send the information about themselves and their neighbors, an identifying code is necessary. However, the systems where the sensors work without failure or if their only task is to test their neighborhoods (not themselves) then we shall search for locating-dominating sets. Moreover, the existence of identifying codes is not always guaranteed in a graph (as we shall see in our later discussion) and then a locating-dominating set is the next best alternative.

In this paper, we study the locating-dominating sets and identifying codes for the graph associated to finite vector space as defined in [16]. Also, we find location-domination number and identifying number of the graph and study the exchange property of the graph for these graph invariants.

Now, we recall some definitions of graph theory which are necessary for this article. We use  $\Gamma$  to denote a connected graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . The *degree*

of the vertex  $v$  in  $\Gamma$ , denoted by  $\deg(v)$ , is the number of edges to which  $v$  belongs. The *open neighborhood* of the vertex  $u$  of  $\Gamma$  is  $N(u) = \{v \in V(\Gamma) : uv \in E(\Gamma)\}$  and the *closed neighborhood* of  $u$  is  $N[u] = N(u) \cup \{u\}$ .

Formally, we define a locating-dominating set as: A set  $L_D$  of vertices of  $\Gamma$  is called a locating-dominating set for  $\Gamma$  if for every two distinct vertices  $u, v \in V(\Gamma) \setminus L_D$ , we have  $\emptyset \neq N(u) \cap L_D \neq N(v) \cap L_D \neq \emptyset$ . The *location-domination number*, denoted by  $\lambda(\Gamma)$ , is the minimum cardinality of a locating-dominating set of  $\Gamma$ .

An identifying code is a subset of vertices in a graph with the property that the neighborhood of every vertex has a unique intersection with the code. Formally it is defined as: A set  $I_D$  is called an *identifying code* for the graph  $\Gamma$  if  $N[u] \cap I_D \neq N[v] \cap I_D$  for all  $u, v \in V(\Gamma)$ . The cardinality of a smallest identifying code is called the *identifying number* of  $\Gamma$  and we denote it by  $I(\Gamma)$ .

Unlike identifying codes, every graph has a trivial locating-dominating set, the entire set of vertices. On the other hand, a graph may not be an identifying code, because if  $N[u] = N[v]$  for some  $u, v \in V(\Gamma)$ , then clearly  $V(\Gamma)$  is not an identifying code. Since an identifying code is also a locating-dominating set, therefore

$$\lambda(\Gamma(\mathbb{V})) \leq I(\Gamma(\mathbb{V})). \quad (1)$$

Two vertices  $u, v$  are *adjacent twins* if  $N[u] = N[v]$  and *non-adjacent twins* if  $N(u) = N(v)$ . If  $u, v$  are adjacent or non-adjacent twins, then  $u, v$  are *twins*. A set of vertices  $T$  is called a *twin-set* if any two of its vertices are twins [20]. By definition of twin vertices and twin-set, we have the following straightforward results:

**Proposition 1.1.** *Suppose that  $u, v$  are twins in a connected graph  $\Gamma$  and  $L_D$  is a locating-dominating set of  $\Gamma$ , then either  $u$  or  $v$  is in  $L_D$ . Moreover, if  $u \in L_D$  and  $v \notin L_D$ , then  $(L_D \setminus \{u\}) \cup \{v\}$  is a locating-dominating set of  $\Gamma$ .*

**Proposition 1.2.** *Let  $T$  be a twin-set of order  $m \geq 2$  in a connected graph  $\Gamma$ . Then, every locating-dominating set  $L_D$  of  $\Gamma$  contains at least  $m - 1$  vertices of  $T$ .*

## 1.1 Non-Zero Component Graph

Let  $\mathbb{V}$  be a vector space over a field  $\mathbb{F}$  with a basis  $\{b_1, b_2, \dots, b_n\}$ . A vector  $v \in \mathbb{V}$  is expressed uniquely as a linear combination of the form  $v = c_1b_1 + c_2b_2 + \dots + c_nb_n$ . A *non-zero component graph*, denoted by  $\Gamma(\mathbb{V})$ , can be associated with a finite dimensional vector space in the following way: the vertex set of the graph  $\Gamma(\mathbb{V})$  consists of the non-zero vectors and two vertices are joined by an edge if they share at least one  $b_i$  with non-zero coefficient in their unique linear combination with respect to  $\{b_1, b_2, \dots, b_n\}$  [16]. It is proved in [16] that  $\Gamma(\mathbb{V})$  is independent of the choice of basis, i.e., isomorphic non-zero component graphs are obtained for two different bases.

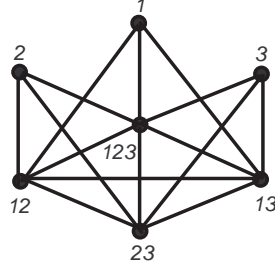


Figure 1:  $\dim(\mathbb{V}) = 3$ ;  $\mathbb{F} = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ .

**Theorem 1.3.** [17] *If  $\mathbb{V}$  be an  $n$ -dimensional vector space over a finite field  $\mathbb{F}$  with  $q$  elements, then the order of  $\Gamma(\mathbb{V})$  is  $q^n - 1$  and the size of  $\Gamma(\mathbb{V})$  is*

$$\frac{q^{2n} - q^n + 1 - (2q - 1)^n}{2}.$$

**Theorem 1.4.** [16] *Let  $\mathbb{V}$  be an  $n$ -dimensional vector space over a finite field  $F$  with  $q$  elements and  $\Gamma(\mathbb{V})$  be its associated graph with respect to a basis  $\{b_1, b_2, \dots, b_n\}$ , then a vertex having  $s$  non-zero coefficients in its unique linear combination of basis vector has degree  $(q^s - 1)q^{n-s} - 1$ .*

## 2 Locating-Dominating Sets and Identifying Codes of Non-Zero Component Graph

In this section, we study the location-domination number of non-zero component graph  $\Gamma(\mathbb{V})$ .

We partition the vertex set of  $\Gamma(\mathbb{V})$  into  $n$  classes  $T_i$ , where  $T_i = \{v \in \mathbb{V} : v \text{ is a linear combination of basis vectors with } i \text{ non-zero coefficients}\}$ . For example, if  $n = 3$  and  $q = 2$ , then  $T_2 = \{b_1 + b_2, b_2 + b_3, b_1 + b_3\}$ .

**Lemma 2.1.** *Let  $\mathbb{V}$  be a vector space of dimension  $n$  over a field  $\mathbb{F}$  of 2 elements. If  $v \in T_s$  for  $s$  ( $1 \leq s \leq n$ ), then for  $r$  ( $1 \leq r \leq n$ )*

$$|N(v) \cap T_r| = \begin{cases} \binom{n}{r} - \binom{n-s}{r} - 1 & \text{if } r \leq n-s \text{ and } r = s \\ \binom{n}{r} - \binom{n-s}{r} & \text{if } r \leq n-s \text{ and } r \neq s \\ \binom{n}{r} - 1 & \text{if } n-s < r \leq n \text{ and } r = s \\ \binom{n}{r} & \text{if } n-s < r \leq n \text{ and } r \neq s. \end{cases}$$

*Proof.* We consider the following cases for  $r$ :

1. If  $r \leq n - s$ , then  $\binom{n-s}{r}$  elements of  $T_r$  have  $s$  zero coefficients in their unique linear combination of basis vectors for those  $s$  basis vectors which have the non-zero coefficients in the unique linear combination of  $v$ , and hence these elements of  $T_r$  are not adjacent to  $v$ . Thus,  $|N(v) \cap T_r| = \binom{n}{r} - \binom{n-s}{r}$  or  $\binom{n}{r} - \binom{n-s}{r} - 1$  according as  $r \neq s$  or  $r = s$ , respectively.
2. If  $r > n - s$ , then each element of  $T_r$  will have at least one non-zero coefficient in its unique linear combination of basis vectors for those  $s$  basis vectors which have the non-zero coefficients in the unique linear combination of  $v$ , and hence  $v$  is adjacent to all elements of  $T_r$ . Thus,  $|N(v) \cap T_r| = \binom{n}{r}$  or  $\binom{n}{r} - 1$  according as  $r \neq s$  or  $r = s$ , respectively.

□

Let  $v \in T_s$ , then it can be seen from Lemma 2.1 that  $\deg(v) = [\sum_{r=1}^n |N(v) \cap T_r|] - 1 = \sum_{r=1}^{n-s} [\binom{n}{r} - \binom{n-s}{r}] + \sum_{r=n-s+1}^n \binom{n}{r} - 1 = (2^s - 1)2^{n-s} - 1$  which is consistent with Theorem 1.4 for  $q = 2$ .

**Remark 2.2.** Let  $\mathbb{V}$  be a vector space of dimension  $n$  over a field  $\mathbb{F}$  of 2 elements. If  $v \in T_s$  for  $s$  ( $1 \leq s \leq n$ ), then  $\deg(v) = (2^s - 1)2^{n-s} - 1$ .

**Lemma 2.3.** Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 4$  over a field  $\mathbb{F}$  of 2 elements. If  $u, v \in V(\Gamma(\mathbb{V})) \setminus T_{n-1}$ , then  $N(u) \cap T_2 \neq N(v) \cap T_2$ .

*Proof.* Since  $u \in T_r$  and  $v \in T_s$  for some  $1 \leq r, s \leq n$  ( $r, s \neq n - 1$ ), therefore  $u$  has  $r$  non-zero coefficients in its unique linear combination of basis vectors  $B = \{b_1, b_2, \dots, b_n\}$ . Let  $B_u \subseteq B$  and  $B_v \subseteq B$  is the set of those basis vectors which has non-zero coefficients in the unique linear combination of basis vectors for  $u$  and  $v$  respectively. Then  $u$  is not adjacent to  $\binom{n-r}{2}$  elements of  $T_2$  which have exactly two non-zero coefficients of basis vectors in  $B_u$  and zero coefficients of basis vectors in  $B \setminus B_u$ . Since  $u \notin T_{n-1}$ , therefore such elements exist in  $T_2$  which has exactly two non-zero coefficients of basis vectors of  $B_u$ . Thus,  $N(u) \cap T_2 = T_2 \setminus \{\text{two element sum of basis vectors in } B \setminus B_r\}$ . Since  $u \neq v$ , therefore  $B_u \neq B_v$ , and hence  $N(u) \cap T_2 \neq N(v) \cap T_2$ . □

An immediate consequence of Lemma 2.3 is that the set  $T_2 \cup T_{n-1}$  forms a locating-dominating set for  $\Gamma(\mathbb{V})$  for a vector space  $\mathbb{V}$  of dimension  $n \geq 4$  over a field of 2 elements.

Since elements of  $T_{n-1}$  have non-zero coefficients for  $n-1$  basis vectors, therefore we use the notation  $u_j = \sum_{i=1}^n b_i - b_j$  in proof of Lemma 2.4 for the element of  $T_{n-1}$  which has zero coefficient for the basis vector  $b_j$ . Also,  $N[u_j] = V(\Gamma(\mathbb{V})) \setminus \{b_j\}$ , therefore two elements  $u_i, u_j \in T_{n-1}$  have same neighbors in  $\Gamma(\mathbb{V})$  except the elements  $b_i$  and  $b_j$  of  $T_1$ .

**Lemma 2.4.** *Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 3$  over a field  $\mathbb{F}$  of 2 elements. Let  $L_D$  be a locating-dominating set for  $\Gamma(\mathbb{V})$  and  $|L_D \cap T_1| = s$*

- (a) *If  $0 \leq s \leq n-2$ , then  $|L_D \cap T_{n-1}| \geq n-s$ .*
- (b) *If  $s = n-1$ , then  $|L_D \cap \{T_n \cup T_{n-1}\}| \geq 1$ .*

*Proof.* Without loss of generality assume that  $L_D \cap T_1 = \{b_1, b_2, \dots, b_s\}$ .

- (a) Let  $u_i, u_j \in T_{n-1}$  for  $s+1 \leq i \neq j \leq n$  be two distinct elements of  $T_{n-1}$ , then  $N(u_i) \cap \{L_D \cap T_1\} = N(u_j) \cap \{L_D \cap T_1\} = \emptyset$ . Since  $u_i$  and  $u_j$  have different neighbors only in  $\{b_{s+1}, b_{s+2}, \dots, b_n\} \subseteq T_1$  which is not subset of  $L_D$ , therefore these  $n-s$  elements of  $T_{n-1}$  must belong to  $L_D$ . Hence,  $|L_D \cap T_{n-1}| \geq n-s$ .
- (b) Let  $u_n \in T_{n-1}$  and  $v \in T_n$ , then  $N(u_n) \cap \{L_D \cap T_1\} = N(v) \cap \{L_D \cap T_1\} = \emptyset$ . Since  $u_n$  and  $v$  have only one different neighbor  $b_n \in T_1$  which is not in  $L_D$ , therefore either  $u_n$  or  $v$  must belong to  $L_D$ .  $\square$

**Corollary 2.5.** *Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 3$  over a field  $\mathbb{F}$  of 2 elements. Let  $L_D$  be a locating-dominating set for  $\Gamma(\mathbb{V})$ , then  $|L_D| \geq n$ .*

*Proof.* If  $0 \leq s \leq n-2$ , then  $|L_D \cap \{T_1 \cup T_{n-1}\}| \geq s + n - s = n$  by Lemma 2.4(a). If  $s = n-1$ , then  $|L_D \cap \{T_1 \cup T_{n-1} \cup T_n\}| \geq n-1 + 1 = n$  by Lemma 2.4(b). If  $s = n$ , then clearly  $|L_D \cap T_1| = n$ .  $\square$

Since  $\lambda(P_3) = 2$  where  $P_3$  is the path graph of order 3, therefore we have the following proposition.

**Proposition 2.6.** *Let  $\mathbb{V}$  be a vector space of dimension 2 over a field  $\mathbb{F}$  of 2 elements, then  $\lambda(\Gamma(\mathbb{V})) = 2$ .*

Let  $\mathbb{V}$  be a vector space of dimension  $n$  and  $q \geq 3$ . Then the class  $T_i$  for each  $i$  ( $1 \leq i \leq n$ ) has  $\binom{n}{i}$  twin subsets of vertices of  $\Gamma(\mathbb{V})$  and each of these twin subsets has the cardinality  $(q-1)^i$ . We use the notation  $T_{i_k}$  where  $1 \leq k \leq \binom{n}{i}$  to denote the  $k$ th twin set in the class  $T_i$ .

**Theorem 2.7.** *Let  $\mathbb{V}$  be a vector space over a field  $\mathbb{F}$  of  $q$  elements with  $\{b_1, b_2, \dots, b_n\}$  as basis:*

- a) *If  $q = 2$  and  $n \geq 3$ , then  $\lambda(\Gamma(\mathbb{V})) = n$ .*
- b) *If  $q \geq 3$ , then  $\lambda(\Gamma(\mathbb{V})) = \sum_{i=1}^n \binom{n}{i} ((q-1)^i - 1)$ .*

*Proof.* a) For  $q = 2$  and  $n \geq 3$ , we first prove that  $T_1$  is a locating-dominating set for  $\Gamma(\mathbb{V})$ . Let  $u, v \in V(\Gamma(\mathbb{V})) \setminus T_1$ . If  $u, v \in T_s$  for some  $s$  when  $2 \leq s \leq n-1$ , then both  $u$  and  $v$  have  $s$  non-zero coefficients in their linear combinations of basis vectors. Since  $u \neq v$  and  $s < n$ , therefore  $\emptyset \neq N(u) \cap T_1 \neq N(v) \cap T_1 \neq \emptyset$ . If  $u \in T_r$  and  $v \in T_s$  for some  $r, s$   $2 \leq r \neq s \leq n$ , then  $|N(u) \cap T_1| \neq |N(v) \cap T_1|$  by Lemma 2.1, and hence  $\emptyset \neq N(u) \cap T_1 \neq N(v) \cap T_1 \neq \emptyset$ . Thus,  $T_1$  is a locating dominating set for  $\Gamma(\mathbb{V})$ . Hence,  $\lambda(\Gamma(\mathbb{V})) \leq n$ . Also  $\lambda(\Gamma(\mathbb{V})) \geq n$  by Corollary 2.5.

b) If  $q \geq 3$ , then from Proposition 1.2, a minimal locating-dominating set of  $\Gamma(\mathbb{V})$  contains at least  $(q-1)^i - 1$  vertices from  $T_{i_k}$  for each  $i$  ( $1 \leq i \leq n$ ) and each  $k$  ( $1 \leq k \leq \binom{n}{i}$ ), and hence  $\lambda(\Gamma(\mathbb{V})) \geq \sum_{i=1}^n \binom{n}{i} ((q-1)^i - 1)$ . Moreover, a subset of  $\Gamma(\mathbb{V})$  of cardinality greater than  $\sum_{i=1}^n \binom{n}{i} ((q-1)^i - 1)$  has all the vertices of at least one twin subset  $T_{i_k}$ . Thus, from Proposition 1.1, a locating-dominating set of cardinality greater than  $\sum_{i=1}^n \binom{n}{i} [(q-1)^i - 1]$  is not a minimal locating-dominating set, and hence  $\lambda(\Gamma(\mathbb{V})) \leq \sum_{i=1}^n \binom{n}{i} [(q-1)^i - 1]$ .  $\square$

Since  $I(P_3) = 2$ , therefore we have the following proposition.

**Proposition 2.8.** *Let  $\mathbb{V}$  be a vector space of dimension 2 over a field  $\mathbb{F}$  of 2 elements, then  $I(\Gamma(\mathbb{V})) = 2$ .*

Following theorem gives the identifying number of  $\Gamma(\mathbb{V})$ .

**Theorem 2.9.** *Let  $\mathbb{V}$  be a finite vector space over a field  $\mathbb{F}$  of 2 elements, then  $I(\Gamma(\mathbb{V})) = n$ .*

*Proof.* For  $n \geq 3$  and  $q = 2$ , by Theorem 2.7(a) and inequality (1),  $I(\Gamma(\mathbb{V})) \geq n$ . Note that,  $T_1$  is an identifying code for  $\Gamma(\mathbb{V})$  because for each vertex say  $u \in V(\Gamma(\mathbb{V}))$ ,  $N[u] \cap T_1$  is the set of all those elements of  $T_1$  which has non-zero coefficients in the representation of  $u$  as the unique linear combination of basis vectors. Thus, for any two distinct elements  $u, v \in V(\Gamma(\mathbb{V}))$ ,  $N[u] \cap T_1$  and  $N[v] \cap T_1$  are distinct. Hence,  $I(\Gamma(\mathbb{V})) \leq n$ .  $\square$

Let  $\mathbb{V}$  be a finite vector space and  $q \geq 3$ , then  $\Gamma(\mathbb{V})$  has twin sets  $T_{i_k}$  ( $1 \leq i \leq n$ ) ( $1 \leq k \leq \binom{n}{i}$ ) and each of these twin subset has adjacent twins, therefore identifying code for  $\Gamma(\mathbb{V})$  does not exist. Thus, we have following remark.

**Remark 2.10.** *Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 3$  and  $q \geq 3$ , then identifying code for  $\Gamma(\mathbb{V})$  does not exist.*

**Lemma 2.11.** *Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 3$  and  $q = 2$ , then  $T_1$  is the only minimal identifying code for  $\Gamma(\mathbb{V})$ .*

*Proof.* Suppose on contrary  $I'_D$  be another minimal identifying code of  $\Gamma(\mathbb{V})$ , then there exist at least one element say  $b_r \in T_1$  such that  $b_r \notin I'_D$  (because otherwise  $T_1 \subset I'_D$ ). Take two elements  $u_r \in T_{n-1}$  (using same notation as in proof of Lemma 2.4) and  $w \in T_n$ . Since  $N[w] = V(\Gamma(\mathbb{V}))$  and  $N[u_r] = V(\Gamma(\mathbb{V})) \setminus \{b_r\}$ , therefore  $N[w] \cap I'_D = N[u_r] \cap I'_D \neq \emptyset$ , a contradiction.  $\square$

## 2.1 Exchange Property

Locating-dominating sets are said to have *the exchange property* in a graph  $\Gamma$  if whenever  $L_{D_1}$  and  $L_{D_2}$  are minimal locating-dominating sets for  $\Gamma$  and  $u_1 \in L_{D_1}$ , then there exists  $u_2 \in L_{D_2}$  so that  $(L_{D_2} \setminus \{u_2\}) \cup \{u_1\}$  is also a minimal locating-dominating set. If a graph  $\Gamma$  has the exchange property, then every minimal locating-dominating set for  $\Gamma$  has the same number of vertices. To show that the exchange property does not hold in a graph, it is sufficient to show that there exist two minimal locating-dominating of different cardinalities. However, the condition is not necessary, i.e., the exchange property does not hold and, hence, does not imply that there are locating-dominating sets of different cardinalities.

**Lemma 2.12.** *For  $q = 2$  and  $n > 3$ , the exchange property does not hold for locating-dominating sets in graph  $\Gamma(\mathbb{V})$ .*

*Proof.* For  $n = 4$ , the exchange property does not hold because  $T_1$  and  $\{b_1 + b_4, b_2 + b_4, b_3 + b_4\} \cup T_3$  are minimal locating-dominating sets of different cardinalities. For  $n \geq 5$ ,  $T_1$  and  $T_2 \cup T_{n-1}$  are two locating-dominating sets of cardinalities  $n$  and  $\binom{n}{2} + n$  by Lemma 2.3. For notational convenience, we use  $A = T_2 \cup T_{n-1}$ . We will prove that  $A$  is a minimal locating-dominating set of  $\Gamma(\mathbb{V})$ . Let  $u \in A$  and  $w \in T_n$ . There are two possible cases for  $u$ .

1. If  $u \in T_2$ , then  $u$  has exactly two non-zero coefficients in its unique linear combination of basis vectors, say these vectors set as  $B_u$ . Choose an element in  $v \in T_{n-2}$  such that  $v$  has exactly  $n - 2$  non-zero coefficients in the unique linear combination of basis vectors in  $B \setminus B_u$ . Then  $N(v) \cap A \setminus \{u\} = N(w) \cap A \setminus \{u\}$ . Thus,  $A \setminus \{u\}$  is not locating-dominating.
2. If  $u \in T_{n-1}$ , then  $N(u) \cap A \setminus \{u\} = N(w) \cap A \setminus \{u\}$ .

Thus,  $T_2 \cup T_{n-1}$  is a minimal locating-dominating set. Hence, exchange property does not hold for locating-dominating sets in graph  $\Gamma(\mathbb{V})$ .  $\square$



In the proof of Lemma 2.13, we use the same notation  $T_{i_k}$  for the  $k$ th twin set of class  $T_i$  as we have used in the proof of Theorem 2.7(b).

**Lemma 2.13.** *For  $q \geq 3$ , the exchange property holds for locating-dominating sets in graph  $\Gamma(\mathbb{V})$ .*

*Proof.* Since there are  $(q-1)^i$  choices for removing one vertex from a twin set  $T_{i_k}$  of cardinality  $(q-1)^i$ , therefore there are  $\prod_{i=1}^n \binom{n}{i} (q-1)^i$  minimal locating-dominating sets in  $\Gamma(\mathbb{V})$ . Let  $L_{D_1} \neq L_{D_2}$  be two such minimal locating-dominating sets. Let  $u_1 \in L_{D_1}$ , we further assume that  $u_1 \notin L_{D_2}$  (for otherwise  $(L_{D_2} \setminus \{u_1\}) \cup \{u_1\}$  is, obviously, a minimal locating-dominating set of  $\Gamma(\mathbb{V})$ ). Also,  $u_1 \in T_{i_k}$  for some  $i$  ( $1 \leq i \leq n$ ) and some  $k$  ( $1 \leq k \leq \binom{n}{i}$ ). Since  $u_1 \in \{L_{D_1} \cap T_{i_k}\} \setminus \{L_{D_2} \cap T_{i_k}\}$  and  $L_{D_1}$  and  $L_{D_2}$  are minimal, therefore there exists an element  $u_2 \in \{L_{D_2} \cap T_{i_k}\} \setminus \{L_{D_1} \cap T_{i_k}\}$ . Since both  $u_1$  and  $u_2$  belong to the same twin set  $T_{i_k}$ , therefore by Proposition 1.1  $(L_{D_2} \setminus \{u_2\}) \cup \{u_1\}$  is a minimal locating-dominating set of  $\Gamma(\mathbb{V})$ . Hence, exchange property holds in  $\Gamma(\mathbb{V})$   $\square$

From Lemma 2.11 we have the following remark.

**Remark 2.14.** *Let  $\mathbb{V}$  be a vector space of dimension  $n \geq 3$  and  $q = 2$ , then exchange property holds for identifying code holds in  $\Gamma(\mathbb{V})$ .*

From Lemma 2.12 we have the following remark.

**Remark 2.15.** *The locating-dominating sets does not have the exchange property for all graphs.*

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